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by M. Longnecker and R. J. Serfling

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ABSTRACT

MOMENT INEQUALITIES FOR S_n UNDER GENERAL DEPENDENCE RESTRICTIONS, WITH APPLICATIONS

Consider the sum $S_n = \sum_{k=1}^{n} c_k X_k$, where $\{X_i\}$ is a sequence of random variables and $\{c_i\}$ a sequence of constants. This paper establishes moment inequalities of the form $E\{S_n^\nu\} \le A(\sum_{k=1}^{n} b_k^r c_k^r)^{\nu/r}$, where ν is an even integer, $b_k = E\{X_k^\nu\}$ $(k=1,\ldots,n)$, and A is a constant depending upon ν and the dependence restrictions imposed upon the $\{X_i\}$ but not depending upon the $\{c_i\}$. A further inequality of more complicated form is also established. The dependence restrictions considered are either of the weak multiplicative type or of related types, namely exchangeable sequences and strongly mixing sequences. Three applications are developed. One treats the almost sure convergence of $\sum_{k=1}^{\infty} c_k X_k$, under mild dependence restrictions and the condition $\sum_{k=1}^{\infty} c_k^2 < \infty$. Secondly, an improved technique is presented for the problem of establishing the rate of convergence in the central limit theorem for simple linear rank statistics. Finally, the central limit theorem for strongly mixing summands is treated.

1. Introduction. For random variables $X_1, X_2, \ldots$ and constants $c_1, c_2 \ldots$, put $S_n = \sum_1^n c_k X_k$. This paper establishes moment inequalities for S_n, i.e., upper bounds on the moments $E\{S_n^\nu\}$, for even integer ν and under suitable restrictions on the dependence structure of the sequence $\{X_i\}$. In most cases, the bound is of the form $A(\sum_{k=1}^n b_k^r |c_k|^r)^{\nu/r}$, where $b_k = E\{X_k^\nu\}$ and A is a constant depending upon ν and the dependence restrictions but not upon the c_k's. In the exceptional cases, the bound is of a more complex configuration.

The types of dependence restrictions considered are either of the weak multiplicative type or else of related types, namely exchangeable sequences and strongly mixing sequences.

The various results are obtained by use of a single general technique of bounding $E\{S_n^\nu\}$. The variable S_n^ν is expressed as the sum of two terms Z_ν and T_ν, where T_ν is the summation of $c_{i_1} \cdots c_{i_\nu} X_{i_1} \cdots X_{i_\nu}$ over all choices of $i_1 < \cdots < i_\nu$ from $\{1, \ldots, n\}$. A bound is placed on $E\{Z_\nu\}$ without any restriction on the dependence structure of the X_i's. For each of the dependence restrictions considered, a corresponding bound is placed on $|E\{T_\nu\}|$ and this bound is then combined with the bound on $E\{Z_\nu\}$ to yield the desired moment inequality.

In Section 2, several dependence restrictions of the weak multiplicative type are introduced. The term "weak multiplicative" refers to any form of restriction on the product moments $E\{X_{i_1} X_{i_2} \cdots X_{i_\nu}\}$ of order ν. Two of the conditions can be characterized as orthogonality-related dependence restrictions. A third condition is similar in form, but is motivated in a different way by examining the structure of the product moments of a Gaussian sequence. Also, a "product-moment exchangeable" restriction is formulated. Finally, the structure of the product moments of a strongly mixing sequence are considered.

Section 3 treats the fundamental decomposition of $(\sum_1^n c_k X_k)^\nu$ which is used in Section 4 to obtain moment inequalities for S_n^ν. In Section 4, an upper bound for $E\{S_n^\nu\}$ is derived for each of the dependence restrictions discussed in Section 2.

Section 5 presents important applications of the inequalities derived in Section 4. In conjunction with a maximal inequality of Longnecker and Serfling (1976), the almost sure convergence of $\sum_1^\infty c_k X_k$ under mild dependence restrictions and the condition $\sum_1^\infty c_k^2 < \infty$ is established. Next, one of the moment inequalities is used to improve a technique of Puri and Jurěcková (1975) in obtaining the rate of convergence in a central limit theorem for simple linear rank statistics. A third application utilizes one of the inequalities to obtain a central limit result for sums of the form $\sum_1^n f(X_i)$, where f is a bounded function and the X_i's are strongly m-"ing.

2. Dependence restrictions of weak multiplicative type. Several alternative dependence restrictions of general scope are formulated here. For each of these conditions, a moment inequality for S_n is derived in Section 4.

DEFINITION. A sequence of random variables $\{X_i\}$ satisfies *Condition A* with respect to an even integer ν, a sequence of constants $\{a_i\}$, and a symmetric function g of $\nu-1$ arguments if

(2.1a) $$|E\{X_{i_1} \cdots X_{i_\nu}\}| \le g(i_2-i_1, i_3-i_2, \cdots, i_\nu-i_{\nu-1})a_{i_1}\cdots a_{i_\nu}$$

for all $1 \le i_1 < \cdots < i_\nu$, and if

(2.1b) $$\sum_{k=1}^\infty \sum_{j_1=1}^k \cdots \sum_{j_{\nu-2}=1}^k g(j_1, \cdots, j_{\nu-2}, k) < \infty . \ \square$$

DEFINITION. A sequence of random variables $\{X_i\}$ satisfies *Condition B* with respect to an even integer ν, a sequence of constants $\{a_i\}$, and a symmetric function

g of $\frac{1}{2}\nu$ arguments if

$$(2.2a)\qquad |E\{X_{i_1} \cdots X_{i_\nu}\}| \le g(i_2-i_1, i_4-i_3, \cdots, i_\nu-i_{\nu-1})a_{i_1}\cdots a_{i_\nu}$$

for all $1 \le i_1 < \cdots < i_\nu$, and if

$$(2.2b)\qquad \sum_{k=1}^{\infty} \sum_{j_1=1}^{k} \cdots \sum_{j_{\frac{1}{2}\nu-1}=1}^{k} g(j_1, \cdots, j_{\frac{1}{2}\nu-1}, k) < \infty . \ \square$$

For the case $\nu=2$, Conditions A and B coincide and represent a simple relaxation of orthogonality. This case also includes the notion of quasi-orthogonality treated in Kac, Salem and Zygmund (1948). For $\nu\ge 4$, Conditions A and B are considerably more powerful then orthogonality (although not implying orthogonality). See further discussion in Section 5. Moment inequalities for S_n under Conditions A and B are provided in Section 4, Theorems 4.3, 4.3*, 4.5 and Corollary 4.4.

Two specialized forms of Condition B are now presented.

DEFINITION. A sequence $\{X_i\}$ satisfies *Condition B1* with respect to an even integer ν, a sequence of constants $\{a_i\}$, and a function $f(j)$ if

$$(2.3a)\qquad |E\{X_{i_1} \cdots X_{i_\nu}\}| \le \min\{f(i_2-i_1), f(i_4-i_3),\cdots,f(i_\nu-i_{\nu-1})\}a_{i_1}\cdots a_{i_\nu}$$

for all $1 \le i_1 <\cdots< i_\nu$, and if

$$(2.3b)\qquad \sum_{j=1}^{\infty} j^{\frac{1}{2}\nu-1} f(j) < \infty . \ \square$$

With $g(j_1,\cdots,j_{\frac{1}{2}\nu}) = \min\{f(j_1),\cdots,f(j_{\frac{1}{2}\nu})\}$, (2.2a) and (2.3a) are equivalent and (2.3b) implies (2.2b). The case $\nu=4$ of (2.3a) is included in a set of conditions introduced and utilized by Révész (1969). In general form, Condition B1 has been used by Gapoškin (1972).

DEFINITION. A sequence $\{X_i\}$ satisfies *Condition B2* with respect to an even integer ν, constants $\{a_i\}$, and a function $f(j)$ if

$$(2.4a) \qquad |E\{X_{i_1} \cdots X_{i_\nu}\}| \le f(i_2-i_1)f(i_4-i_3)\cdots f(i_\nu-i_{\nu-1})a_{i_1}\cdots a_{i_\nu}$$

for all $1 \le i_1 < \cdots < i_\nu$, and if

$$(2.4b) \qquad \sum_{j=1}^{\infty} f(j) < \infty. \quad \square$$

Note that (2.4a) is stronger than (2.3a), while (2.4b) is weaker than (2.3b). Also, with $g(j_1,\cdots,j_{\frac{1}{2}\nu}) = f(j_1)\cdots f(j_{\frac{1}{2}\nu})$, (2.2a) and (2.4a) are equivalent and (2.4b) implies (2.2b). Further, with $g(j_1,\cdots,j_{\nu-1}) = f(j_1)f(j_3)\cdots f(j_{\nu-1})$, (2.1a) and (2.4a) are equivalent and (2.3b) implies (2.1b). Thus a sequence satisfying both Conditions B1 and B2 also satisfies Condition A. Moment inequalities for S_n under Conditions B1 and B2 are given in Corollaries 4.6 and 4.7.

Conditions A, B, B1 and B2 are seemingly of the character of orthogonality-related dependence restrictions. But also they are closely related to a dependence restriction which has arisen in the quite different context of time series analysis, with particular reference to Gaussian time series. Before stating the condition, we consider the well-known fact (Anderson (1958), page 39) that for $(X_{i_1},X_{i_2},X_{i_3},X_{i_4})$ multivariate normal with mean vector 0, the product moment of order 4 is given by

$$E\{X_{i_1}X_{i_2}X_{i_3}X_{i_4}\} = E\{X_{i_1}X_{i_2}\}E\{X_{i_3}X_{i_4}\} + E\{X_{i_1}X_{i_3}\}E\{X_{i_2}X_{i_4}\}$$

$$(2.5) \qquad + E\{X_{i_1}X_{i_4}\}E\{X_{i_2}X_{i_3}\}.$$

If it is further assumed that $\{X_i\}$ is stationary, i.e., $E\{X_iX_j\} = R(j-i)$, and that $|R(k)|$ is nonincreasing, then it follows easily that, for $i_1 \le i_2 \le i_3 \le i_4$,

(2.6) $$|E\{X_{i_1}X_{i_2}X_{i_3}X_{i_4}\}| \le 2|R(i_2-i_1)R(i_4-i_3)| + \min\{|R(i_2-i_1)|,|R(i_4-i_3)|\}|R(i_3-i_2)|.$$

Here the first term is of the form of Condition B2. The second term motivates the following definition.

DEFINITION. A sequence $\{X_i\}$ satisfies *Condition C* with respect to an even integer ν, constants $\{a_i\}$, a function $f(j)$, and a function g of $\frac{1}{2}\nu-1$ arguments if

(2.7a) $$|E\{X_{i_1}\cdots X_{i_\nu}\}| \le \min\{f(i_2-i_1),f(i_\nu-i_{\nu-1})\}g(i_3-i_2,i_5-i_4,\cdots,i_{\nu-1}-i_{\nu-2})a_{i_1}\cdots a_{i_\nu}$$

for all $1 \le i_1 < \cdots < i_\nu$, if

(2.7b) $$\sum_{j=1}^{\infty} f(j) < \infty,$$

and if

(2.7c) $$\sum_{\ell=1}^{\frac{1}{2}\nu-1}\ \sum_{j_\ell=1}^{\infty}\ \sum_{j_1=1}^{j_\ell}\ \sum_{j_2=1}^{j_\ell}\cdots\sum_{j_{\ell-1}=1}^{j_\ell}\ \sum_{j_{\ell+1}=1}^{j_\ell}\cdots\sum_{j_{\frac{1}{2}\nu-1}=1}^{j_\ell} g(j_1,\cdots,j_{\frac{1}{2}\nu-1}) < \infty . \quad \square$$

An associated moment inequality for S_n is provided in Theorem 4.8.

The next type of dependence restriction strengthens in common the product-moment inequality relations of Conditions A and B but avoids imposition of summability requirements. Application in connection with rank statistic problems is discussed in Section 5. An associated moment inequality for S_n is provided in Theorem 4.9.

DEFINITION. A sequence $X_1,\cdots,X_n$ is *product-moment exchangeable* with respect to an even integer ν, constants $\{a_i\}$, and constant $G_{\nu,n}$ if

(2.8) $$E\{X_{i_1}\cdots X_{i_\nu}\} = G_{\nu,n}\, a_{i_1}\cdots a_{i_\nu}$$

for all $1 \le i_1 < \cdots < i_\nu \le n$. □

Finally, *mixing* dependence is considered. Associated moment inequalities are provided in Corollary 4.10 and Theorem 4.11, and applications are discussed in Section 5. As remarked below, in the context of a bounded strictly stationary sequence $\{X_i\}$, mixing dependence is a special case of Condition B1.

DEFINITIONS. Let $\{X_i, i \in I\}$ and $\{X_j; j \in J\}$ be two families of random variables. The *mixing number* measuring the dependence between the two families is $\phi(I;J) = \sup_{A,B}|P(AB)-P(A)P(B)|$, where the ranges of A and B are the σ-fields generated by $\{X_i, i \in I\}$ and $\{X_j, j \in J\}$, respectively. For a strictly stationary sequence $\{X_i\}$, the *Rosenblatt mixing numbers* are given by

$$\phi_n = \phi(\{i: i \le 0\}; \{j: j \ge n\}),\ n = 1,2,\cdots .$$

If $\phi_n \downarrow 0$, the sequence $\{X_i\}$ is called *strongly mixing*. □

REMARK. Let $\{X_i\}$ be strictly stationary with $E\{X_i\} \equiv 0$, *strongly mixing*, and *bounded*: $|X_i| \le C$, all i. By a lemma of Ibragimov (1962), for $i_1 < i_2 < i_3 < i_4$,

(2.9) $$|E\{X_{i_1} X_{i_2} X_{i_3} X_{i_4}\}| \le 4C^4 \min\{\phi_{i_2-i_1}, \phi_{i_4-i_3}\} .$$

Thus, with $\nu=4$, $f(j) = 4\phi_j$ and $a_i \equiv C$, Condition B1 holds if

(2.10) $$\sum_{j=1}^{\infty} j\phi_j < \infty. \ \square$$

3. <u>Preliminary lemmas on products and sums</u>. Two well-known and easily proved numerical inequalities are stated in the following

LEMMA 3.1. *Let* $\{a_i\}$ *be nonnegative constants and let* $0 < p < 1$. *Then*

$$\left(\sum_{i=1}^{n} a_i\right)^p \le \sum_{i=1}^{n} a_i^p \tag{3.1}$$

and

$$\prod_{i=1}^{n} a_i \le \frac{1}{n} \sum_{i=1}^{n} a_i^p . \tag{3.2}$$

The development of Section 4 involves sums of the form

$$T_j = \sum_{k=1}^{n} (c_k X_k)^j , \quad j \ge 1 , \tag{3.3}$$

generated by random variables $X_1,\cdots,X_n$ and constants $c_1,\cdots,c_n$. Put $T_o = 1$. The treatment will make use of the fact that sums of the form

$$\sum\nolimits_{(m)} c_{i_1} \cdots c_{i_m} X_{i_1} \cdots X_{i_m} , \tag{3.4}$$

where $\sum_{(m)}$ denotes summation over all m-tuples $(i_1,\cdots,i_m)$ of distinct integers from the set $\{1,\cdots,n\}$, may be represented in terms of the sums $T_1,\cdots,T_m$, with the representation not depending upon n. For example, the identity

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i \ne j} a_i a_j + \sum_{i=1}^{n} a_i^2$$

yields

$$\sum\nolimits_{(2)} c_{i_1} c_{i_2} X_{i_1} X_{i_2} = T_1^2 - T_2 . \tag{3.5}$$

Likewise it is seen that

$$\sum_{(3)} c_{i_1} c_{i_2} c_{i_3} X_{i_1} X_{i_2} X_{i_3} = T_1^3 - 3T_1T_2 + 2T_3 \ . \tag{3.6}$$

In general, the m-fold symmetric function (3.4) may be handled by reduction to lower cases, as described in Burnside and Panton (1899). Put

$$I_m = \{(i_1,\cdots,i_m)\colon \text{each } i_j \geq 0;\ \text{at least one } i_j \geq 2;\ i_1+\cdots+i_m = m\} \tag{3.7}$$

and denote summation over $(i_1,\cdots,i_m) \in I_m$ by $\sum_{(I_m)}$.

LEMMA 3.2. *There exist integers* $d(i_1,\cdots,i_m)$ *for* $(i_1,\cdots,i_m) \in I_m$ *such that*

$$T_1^m = \sum_{(m)} c_{i_1}\cdots c_{i_m} X_{i_1}\cdots X_{i_m} + \sum_{(I_m)} d(i_1,\cdots,i_m) T_{i_1}\cdots T_{i_m} \ . \tag{3.8}$$

For later reference, define

$$D_m = \sum_{(I_m)} |d(i_1,\cdots,i_m)| \tag{3.9}$$

and note that $D_m \geq 1$ and D_m depends only on m.

The representation given by Lemma 3.2 provides the fundamental decomposition of S_n^ν which is utilized in Section 4. Namely, noting that T_1 and S_n are the same, we have

$$S_n^\nu = W_\nu + Z_\nu \ , \tag{3.10}$$

where

$$W_\nu = \sum_{(\nu)} c_{i_1}\cdots c_{i_\nu} X_{i_1}\cdots X_{i_\nu} \tag{3.11}$$

and

$$Z_\nu = \sum_{(I_\nu)} d(i_1,\cdots,i_\nu) T_{i_1}\cdots T_{i_\nu} \ . \tag{3.12}$$

4. Upper bounds for $E\{S_n^\nu\}$. Based on the decomposition (3.10), upper bounds for $E\{S_n^\nu\}$ will be obtained by combining separate upper bounds for $E\{W_\nu\}$ and $E\{Z_\nu\}$ via

$$E\{S_n^\nu\} = E\{W_\nu\} + E\{Z_\nu\}. \tag{4.1}$$

The following result deals with $E\{Z_\nu\}$ without restriction of the dependence of the X_i's. Previous forms of the result are contained in Serfling (1969), Komlós (1972) and Gapoškin (1972).

LEMMA 4.1. *Let* $X_1,\cdots,X_n$ *satisfy* $E\{X_i^\nu\} < \infty$, $1 \le i \le n$, *for an even integer* ν. *Then there exists an integer* h, $0 \le h \le \nu-2$, *such that*

$$E\{Z_\nu\} \le D_\nu[E\{T_1^\nu\}]^{h/\nu}[E\{T_2^{\nu/2}\}]^{(\nu-h)/\nu}, \tag{4.2}$$

with D_ν *defined by* (3.9).

PROOF. Consider $(i_1,\cdots,i_\nu) \in I_\nu$. Let $t = t(i_1,\cdots,i_\nu)$ denote the number of i_j's equal to 1, and note that $0 \le t \le \nu-2$ must hold. Now observe that, for $j \ge 2$, (3.1) implies

$$|T_j|^{2/j} \le \left(\sum_{k=1}^n |c_kX_k|^j\right)^{2/j} \le \sum_{k=1}^n (c_kX_k)^2 = T_2 ,$$

i.e.,

$$|T_j| \le T_2^{j/2} .$$

Hence

$$|T_{i_1}\cdots T_{i_\nu}| \le |T_1|^t \, T_2^{(\nu-t)/2} , \tag{4.3}$$

and thus, by the Hölder inequality,

$$E\{|T_{i_1}\cdots T_{i_\nu}|\} \le [E\{T_1^\nu\}]^{t/\nu}[E\{T_2^{\nu/2}]^{(\nu-t)/\nu} . \tag{4.4}$$

Now choose h to be the value of t, $0 \le t \le \nu-2$, which maximizes the right-hand side of (4.4). Then (4.2) follows from (4.4) and (3.12). □

Application of Lemma 4.1 will be made through the following corollary, which provides a bound on $E\{Z_\nu\}$ similar in form to a result of Móricz (1976).

COROLLARY 4.2. *Let* $X_1,\cdots,X_n$ *satisfy* $b_i = E\{X_i^\nu\} < \infty$, $1 \le i \le n$, *for an even integer* ν. *Then there exists an integer* h, $0 \le h \le \nu-2$, *such that for* $0 < \gamma \le 2$,

$$(4.5)\qquad E\{Z_\nu\} \le D_\nu[E\{S_n^\nu\}]^{h/\nu}[\sum_{k=1}^n b_k^\gamma |c_k|^\gamma]^{(\nu-h)/\gamma} .$$

PROOF. By the Minkowski inequality and (3.1),

$$[E\{T_2^{\nu/2}\}]^{2/\nu} \le \sum_{k=1}^n b_k^2 c_k^2 \le \left[\sum_{k=1}^n b_k^\gamma |c_k|^\gamma\right]^{2/\gamma} ,$$

so that

$$(4.6)\qquad [E\{T_2^{\nu/2}\}]^{(\nu-h)/\nu} \le \left[\sum_{k=1}^n b_k^\gamma |c_k|^\gamma\right]^{(\nu-h)/\gamma}$$

for h as given in Lemma 4.1. Recall that $S_n = T_1$. Thus (4.5) follows from (4.2) and (4.6). □

The following three results pertain to the dependence restriction Condition A defined in Section 2. Firstly, a moment inequality for S_n is established under a broadened form of Condition A in which the function g is not required to be symmetric and the summability condition (2.1b) is not imposed. Secondly, a useful implication for the case of g symmetric is obtained. Finally, conclusions under the summability condition are given.

In these and several subsequent results, the following notation will be used. For $q \geq 1$, let p be defined by

$$1/p + 1/q = 1 \tag{4.7a}$$

and let γ be defined by

$$\gamma = \begin{cases} 2 & \text{if } q = 1 \\ \min\{\tfrac{1}{2}p,\ q\} & \text{if } q > 1\ . \end{cases} \tag{4.7b}$$

THEOREM 4.3. *Let* $X_1,\cdots,X_n$ *satisfy* $b_i^\nu = E\{X_i^\nu\} < \infty$, $1 \leq i \leq n$, *for an even integer* ν. *Suppose that, for a function* g *of* $\nu-1$ *arguments,*

$$|E\{X_{i_1}\cdots X_{i_\nu}\}| \leq g(i_2-i_1,\ i_3-i_2,\cdots,i_\nu-i_{\nu-1})b_{i_1}\cdots b_{i_\nu} \tag{4.8}$$

for all $1 \leq i_1 <\cdots< i_\nu \leq n$. *Let* $q \geq 1$ *be given. Put*

$$\alpha_{qn} = \left[\sum_{j_1=1}^{n-\nu+1}\ \sum_{j_2=1}^{n-\nu+2-j_1}\cdots\sum_{j_{\nu-1}=1}^{n-1-j_1-\cdots-j_{\nu-2}} g^q(j_1,j_2,\cdots,j_{\nu-1})\right]^{\frac{1}{q}}\ . \tag{4.9}$$

Define γ *by* (4.7) *and* D_ν *by* (3.9). *Then*

$$E\left\{\left[\sum_{k=1}^{n} c_kX_k\right]^\nu\right\} \leq [\nu!\ \alpha_{qn} + D_\nu]^{\nu/2}\left[\sum_{k=1}^{n} b_k^\gamma|c_k|^\gamma\right]^{\nu/\gamma}\ . \tag{4.10}$$

PROOF. First an upper bound on $|E\{W_\nu\}|$ is obtained. Put $d_i = b_i|c_i|$. Take the case $q > 1$. By (3.11) and (4.8), and the use of Hölder's inequality,

$$|E\{W_\nu\}| \le \nu! \sum_{1\le i_1<\cdots<i_\nu\le n} \cdots \sum d_{i_1}\cdots d_{i_\nu}\, g(i_2-i_1,\, i_3-i_2,\cdots, i_\nu-i_{\nu-1})$$

$$\le \nu!\left[\sum_{1\le i_1<\cdots<i_\nu\le n}\cdots\sum d_{i_1}^{\frac{1}{2}p}\cdots d_{i_\nu}^{\frac{1}{2}p}\right]^{\frac{1}{p}}\left[\sum_{1\le i_1<\cdots<i_\nu\le n}\cdots\sum d_{i_1}^{\frac{1}{2}q}\cdots d_{i_\nu}^{\frac{1}{2}q}\, g^q(i_2-i_1,\cdots,i_\nu-i_{\nu-1})\right]^{\frac{1}{q}}$$

$$\le \nu!\left[\sum_{i=1}^{n} d_i^{\frac{1}{2}p}\right]^{\frac{\nu}{p}}\left[\sum_{j_1=1}^{n-\nu+1}\ \sum_{j_2=1}^{n-\nu+2-j_1}\cdots\sum_{j_{\nu-1}=1}^{n-1-j_1-\cdots-j_{\nu-2}}\ \sum_{k=1}^{n-j_1-\cdots-j_{\nu-1}} d_k^{\frac{1}{2}q} d_{k+j_1}^{\frac{1}{2}q}\cdots d_{k+j_1+\cdots+j_{\nu-1}}^{\frac{1}{2}q}\, g^q(j_1,\cdots,j_{\nu-1})\right]^{\frac{1}{q}} . \tag{4.11}$$

By (3.2)

$$\sum_{k=1}^{n-j_1-\cdots-j_{\nu-1}} d_k^{\frac{1}{2}q}\, d_{k+j_1}^{\frac{1}{2}q}\cdots d_{k+j_1+\cdots+j_{\nu-1}}^{\frac{1}{2}q} \le \frac{1}{\nu}\sum_{k=1}^{n-j_1-\cdots-j_{\nu-1}}\ \sum_{i=1}^{\nu} d_{k+j_1+\cdots+j_{\nu-1}}^{\frac{1}{2}q\nu}$$

$$\le \frac{1}{\nu}\sum_{i=1}^{\nu}\sum_{k=1}^{n} d_k^{\frac{1}{2}q\nu} = \sum_{k=1}^{n} d_k^{\frac{1}{2}q\nu} . \tag{4.12}$$

But (3.1) implies

$$\sum_{k=1}^{n} d_k^{\frac{1}{2}q\nu} \le \left(\sum_{k=1}^{n} d_k^q\right)^{\frac{1}{2}\nu} . \tag{4.13}$$

Therefore, by (4.9), (4.11), (4.12) and (4.13),

$$|E\{W_\nu\}| \le \nu!\left[\sum_{k=1}^{n} d_k^{\frac{1}{2}p}\right]^{\frac{\nu}{p}}\left[\sum_{k=1}^{n} d_k^q\right]^{\frac{\nu}{2q}}\alpha_{qn}$$

$$\le \nu!\left[\sum_{k=1}^{n} d_k^{\gamma}\right]^{\nu/\gamma}\alpha_{qn} . \tag{4.14}$$

For the case $q = 1$, a similar argument *without* the use of Hölder's inequality leads to (4.14).

Put $\Delta = \sum_{k=1}^{n} d_k^{\gamma}$. By (4.1), Corollary 4.2 and (4.14),

$$E\{S_n^{\nu}\} \le \nu!\Delta^{\nu/\gamma}\alpha_{qn} + D_{\nu}[E\{S_n^{\nu}\}]^{h/\nu}\Delta^{(\nu-h)/\gamma} \quad , \tag{4.15}$$

where h is an integer satisfying $0 \le h \le \nu-2$.

Suppose that $E\{S_n^{\nu}\} \le \Delta^{\nu/\gamma}$. Then (4.10) holds trivially, since $D_{\nu} \ge 1$. Suppose, on the other hand, that $E\{S_n^{\nu}\} \ge \Delta^{\nu/\gamma}$. Then (4.15) yields

$$E\{S_n^{\nu}\} \le \nu!\Delta^{\nu/\gamma}\, \alpha_{qn}[E\{S_n^{\nu}\}]^{h/\nu}\Delta^{-h/\gamma} + D_{\nu}[E\{S_n^{\nu}\}]^{h/\nu}\Delta^{(\nu-h)/\gamma}$$

$$= [\nu!\, \alpha_{qn} + D_{\nu}][E\{S_n^{\nu}\}]^{h/\nu}\Delta^{(\nu-h)/\gamma} \quad . \tag{4.16}$$

But (4.16) gives

$$E\{S_n^{\nu}\} \le [\nu!\, \alpha_{qn} + D_{\nu}]^{\nu/(\nu-h)}\Delta^{\nu/\gamma}$$

$$\le [\nu!\, \alpha_{qn} + D_{\nu}]^{\nu/2}\Delta^{\nu/\gamma} \quad ,$$

the latter step since $D_{\nu} \ge 1$ and $\nu-h \ge 2$. Thus, in this case also, (4.10) holds. □

THEOREM 4.3*. *Along with the assumptions of Theorem 4.3, suppose that* g *is symmetric, and put*

$$\alpha^*_{qn} = \left[(\nu-1) \sum_{k=1}^{n} \sum_{j_1=1}^{k} \cdots \sum_{j_{\nu-2}=1}^{k} g^q(j_1,\cdots,j_{\nu-2},k)\right]^{\frac{1}{q}} . \tag{4.17}$$

Then

$$E\left\{\left(\sum_{k=1}^{n} c_k X_k\right)^{\nu}\right\} \le [\nu!\, \alpha^*_{qn} + D_{\nu}]^{\nu/2} \left[\sum_{k=1}^{n} b_k^{\gamma}|c_k|^{\gamma}\right]^{\nu/\gamma} . \tag{4.18}$$

PROOF. Define the set J_ℓ by

$$J_\ell = \{(j_1,\cdots,j_{\nu-1}) : 1 \le j_1,\cdots,j_{\nu-1} \le n;\ \sum_{i=1}^{\nu-1} j_i \le n;\ j_\ell = \max\{j_1,\cdots,j_{\nu-1}\}\} ,$$

for $\ell = 1,\cdots,\nu-1$. Then

$$\alpha_{qn}^q \le \sum_{\ell=1}^{\nu-1} \sum_{\substack{j_1=1 \\ (j_1,\cdots,j_{\nu-1}) \in J_\ell}}^{n-\nu+1} \sum_{j_2=1}^{n-\nu+2-j_1} \cdots \sum_{j_{\nu-1}=1}^{n-1-j_1-\cdots-j_{\nu-2}} g^q(j_1,\cdots,j_{\nu-1})$$

$$\le \sum_{\ell=1}^{\nu-1} \sum_{j_\ell=1}^{n} \sum_{j_1=1}^{j_\ell} \cdots \sum_{j_{\ell-1}=1}^{j_\ell} \sum_{j_{\ell+1}=1}^{j_\ell} \cdots \sum_{j_{\nu-1}=1}^{j_\ell} g^q(j_1,\cdots,j_{\nu-1})$$

$$\le (\nu-1) \sum_{k=1}^{n} \sum_{j_1=1}^{k} \cdots \sum_{j_{\nu-2}=1}^{k} g^q(j_1,\cdots,j_{\nu-2},k) = (\alpha_{qn}^*)^q .$$

Thus (4.18) follows by Theorem 4.3. □

Theorems 4.3 and 4.3* apply to a finite sequence $X_1,\cdots,X_n$. For an infinite sequence, the summability part of Condition A becomes relevant. The following result has the useful feature that the upper bound for $E\{S_n^\nu\}$ is of the form $K[\sum_1^n b_k^\gamma |c_k|^\gamma]^{\nu/\gamma}$, where K does not depend on n. This feature is useful in applications such as the question of almost sure convergence of $S_\infty = \sum_1^\infty c_k X_k$.

COROLLARY 4.4. *Suppose that the assumptions of Theorem* 4.3, *for fixed* ν, g *and* q, *hold for all* $n = 1,2,\cdots$. *Put*

$$\alpha_q = \left[\sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \cdots \sum_{j_{\nu-1}=1}^{\infty} g^q(j_1,j_2,\cdots,j_{\nu-1})\right]^{\frac{1}{q}} . \tag{4.19}$$

Then

$$(4.20)\qquad E\left\{\left(\sum_{k=1}^{n} c_k X_k\right)^{\nu}\right\} \le [\nu!\ \alpha_q + D_\nu]^{\nu/2}\left[\sum_{k=1}^{n} b_k^{\gamma}|c_k|^{\gamma}\right]^{\nu/\gamma} .$$

In the case that g *is symmetric, the quantity* α_q *in* (4.20) *may be replaced by*

$$(4.21)\qquad \alpha_q^* = \left[(\nu-1)\sum_{k=1}^{\infty}\sum_{j_1=1}^{k}\cdots\sum_{j_{\nu-2}=1}^{k} g^q(j_1,\cdots,j_{\nu-2}, k)\right]^{\frac{1}{q}} .$$

REMARKS. (i) For asymptotic applications, the most effective choice of q in the use of the preceding result is q = 1, in which case $\gamma = 2$ and (4.20) takes the form

$$(4.22)\qquad E\{S_n^{\nu}\} \le [\nu!\ \alpha_1 + D_\nu]^{\nu/2}\left[\sum_{k=1}^{n} b_k^2 c_k^2\right]^{\nu/2} .$$

Whereas the cases corresponding to q > 1 entail a smaller quantity α_q in place of α_1, the factor

$$(4.23)\qquad \left[\sum_{k=1}^{n} b_k^{\gamma}|c_k|^{\gamma}\right]^{\nu/\gamma}$$

is larger for these cases than for the case q = 1.

(ii) Under related but different conditions, Móricz (1976) establishes, in the proof of his Theorem 1, a result of the form (4.20) for the case that the b_i's are bounded: $b_i \le K < \infty$ (all i). In the role of α_q, Móricz uses

$$(4.24)\qquad m_q = \left[\sum_{1\le i_1<\cdots<i_\nu<\infty}\cdots\sum |E^q\{X_{i_1}\cdots X_{i_\nu}\}|\right]^{1/q} .$$

With

$$g(j_1,j_2,\cdots,j_{\nu-1}) = \sup_i |E\{X_i X_{i+j_1} \cdots X_{i+j_1+\cdots+j_{\nu-1}}\}|,$$

it is seen that $\alpha_q \le m_q$ and hence Corollary 4.4 applies to a larger class of sequences $\{X_i\}$ then covered by Móricz' result. Also, Móricz' results are confined to the case $q \ge 2$. On the other hand, in Móricz' results the factor (4.23) appears with γ replaced by p, which, by (4.7b), yields a sharper factor in the case $q \ge 2$. Note that in this case the most effective choice of q for asymptotic applications is $q = 2$. □

THEOREM 4.5. *Let* $X_1,\cdots,X_n$ *satisfy* $b_i^\nu = E\{X_i^\nu\} < \infty$, $1 \le i \le n$, *for an even integer* ν. *Suppose that, for a symmetric function* g *of* $\frac{1}{2}\nu$ *arguments,*

$$|E\{X_{i_1}\cdots X_{i_\nu}\}| \le g(i_2-i_1,\ i_4-i_3,\cdots,i_\nu-i_{\nu-1})b_{i_1}\cdots b_{i_\nu} \tag{4.25}$$

for all $1 \le i_1 <\cdots< i_\nu \le n$. *Let* $q \ge 1$ *be given. Put*

$$\beta_{qn} = \left[\frac{1}{(\frac{1}{2}\nu-1)!}\sum_{k=1}^{n}\ \sum_{j_1=1}^{k}\cdots\sum_{j_{\frac{1}{2}\nu-1}=1}^{k} g^q(j_1,\cdots,j_{\frac{1}{2}\nu-1},\ k)\right]^{\frac{1}{q}} . \tag{4.26}$$

Define γ *by* (4.7) *and* D_ν *by* (3.9). *Then*

$$E\left\{\left(\sum_{k=1}^{n} c_k X_k\right)^\nu\right\} \le [\nu!\beta_{qn} + D_\nu]^{\nu/2}\left[\sum_{k=1}^{n} b_k^\gamma |c_k|^\gamma\right]^{\nu/\gamma} . \tag{4.27}$$

PROOF. First an upper bound on $|E\{W_\nu\}|$ is obtained. Put $d_i = b_i|c_i|$. Also, denote $\frac{1}{2}\nu$ by m where convenient. Take the case $q > 1$. By (3.11), (4.25), the inequality $2ab \le a^2 + b^2$, and Hölder's inequality,

$$|E\{W_\nu\}| \le \nu! \sum_{1\le i_1<\cdots<i_\nu\le n} \cdots \sum d_{i_1}\cdots d_{i_\nu}\, g(i_2-i_1,\, i_4-i_3,\cdots,i_\nu-i_{\nu-1})$$

$$\le \nu!\left[\sum_{1\le i_1<\cdots<i_\nu\le n}\cdots\sum d_{i_1}^{\frac{1}{2}p}\cdots d_{i_\nu}^{\frac{1}{2}p}\right]^{\frac{1}{p}}\left[\sum_{1\le i_1<\cdots<i_\nu\le n}\cdots\sum d_{i_1}^{\frac{1}{2}q}\cdots d_{i_\nu}^{\frac{1}{2}q}\, g^q(i_2-i_1,\cdots,i_\nu-i_{\nu-1})\right]^{\frac{1}{q}}$$

$$\le \nu!\left[\sum_{k=1}^{n} d_k^{\frac{1}{2}p}\right]^{\frac{\nu}{p}}\left[2^{-m}\sum_{1\le i_1<\cdots<i_\nu\le n}\cdots\sum (d_{i_1}^q+d_{i_2}^q)(d_{i_3}^q+d_{i_4}^q)\cdots(d_{i_{\nu-1}}^q+d_{i_\nu}^q)\times\right.$$

$$\left.\times\, g^q(i_2-i_1,\, i_4-i_3,\cdots,i_\nu-i_{\nu-1})\right]^{\frac{1}{q}}$$

$$\le \nu!\left[\sum_{k=1}^{n} d_k^{\frac{1}{2}p}\right]^{\frac{\nu}{p}}\left[2^{-m}\sum_{1\le i_1<\cdots<i_\nu\le n}\cdots\sum\ \sum_{j_1=1}^{2}\ \sum_{j_2=3}^{4}\cdots\sum_{j_m=\nu-1}^{\nu} d_{i_{j_1}}^q\cdots d_{i_{j_m}}^q\ \times\right.$$

$$\left.\times\, g^q(i_2-i_1,\, i_4-i_3,\cdots,i_\nu-i_{\nu-1})\right]^{\frac{1}{q}}$$

$$\text{(4.28)}\quad \le \nu!\left[\sum_{k=1}^{n} d_k^{\frac{1}{2}p}\right]^{\frac{\nu}{p}}\left[2^{-m}\sum_{j_1=1}^{2}\ \sum_{j_2=3}^{4}\cdots\sum_{j_m=\nu-1}^{\nu} B_{j_1,j_2,\cdots,j_m}\right]^{\frac{1}{q}},$$

where

$$\text{(4.29)}\quad B_{j_1,\cdots,j_m} = \sum_{1\le i_1<\cdots<i_\nu\le n}\cdots\sum d_{i_{j_1}}^q d_{i_{j_2}}^q\cdots d_{i_{j_m}}^q\, g^q(i_2-i_1,\, i_4-i_3,\cdots,i_\nu-i_{\nu-1})\ .$$

As an example of the technique used to place a suitable bound on $B_{j_1,\cdots,j_m}$, consider $B_{1,3,5,\cdots,\nu-1}$. The other $2^m - 1$ terms in (4.29) may be handled in similar fashion. Define

$$J_\ell = \{(i_1,\cdots,i_\nu): 1 \le i_1 <\cdots< i_\nu \le n \text{ and } i_{2\ell}-i_{2\ell-1} = \max\{i_2-i_1,\ i_4-i_3,\cdots,i_\nu-i_{\nu-1}\}\}$$

for $\ell = 1,2,\cdots,m$, and denote summation over $(i_1,\cdots,i_\nu) \in J_\ell$ by $\sum_{(J_\ell)}$. Then

$$B_{1,3,5,\cdots,\nu-1} \le \sum_{\ell=1}^{m} \sum_{(J_\ell)} d_{i_1}^q d_{i_3}^q \cdots d_{i_{\nu-1}}^q g^q(i_2-i_1,i_4-i_3,\cdots,i_\nu-i_{\nu-1})$$

$$= \sum_{\ell=1}^{m} \sum_{i_1=1}^{n-\nu+1} \sum_{i_3=i_1+2}^{n-\nu+3} \cdots \sum_{\substack{i_{\nu-1}=i_{\nu-3}+2 \\ (i_1,\cdots,i_\nu)\,\varepsilon\, J_\ell}}^{n-1} d_{i_1}^q d_{i_3}^q \cdots d_{i_{\nu-1}}^q \times$$

$$\times \sum_{i_2=i_1+1}^{i_3-1} \sum_{i_4=i_3+1}^{i_5-1} \cdots \sum_{\substack{i_\nu=i_{\nu-1}+1 \\ (i_1,\cdots,i_\nu)\,\varepsilon\, J_\ell}}^{n} g^q(i_2-i_1,i_4-i_3,\cdots,i_\nu-i_{\nu-1})$$

$$\le \sum_{\ell=1}^{m} \sum_{1\le i_1<\cdots<i_\nu\le n} \cdots \sum d_{i_1}^q d_{i_3}^q \cdots d_{i_{\nu-1}}^q \times$$

$$\times \sum_{i_{2\ell}-i_{2\ell-1}=1}^{n} \sum_{j_1=1}^{i_{2\ell}-i_{2\ell-1}} \cdots \sum_{j_{\frac12\nu-1}=1}^{i_{2\ell}-i_{2\ell-1}} g^q(j_1,\cdots,j_{\frac12\nu-1},\ i_{2\ell}-i_{2\ell-1})$$

$$\le \frac{1}{(m-1)!}\left[\sum_{k=1}^{n} d_k^q\right]^m \beta_{qn}^q (m-1)! = \left[\sum_{k=1}^{n} d_k^q\right]^m \beta_{qn}^q .$$

It thus follows that

$$|E\{W_\nu\}| \le \nu! \left[\sum_{k=1}^{n} d_k^{\frac12 p}\right]^{\frac{\nu}{p}} \left[\sum_{k=1}^{n} d_k^q\right]^{\frac{\nu}{2q}} \beta_{qn}$$

$$\le \nu! \left[\sum_{k=1}^{n} d_k^\gamma\right]^{\nu/\gamma} \beta_{qn} . \tag{4.30}$$

For the case $q = 1$, a similar argument *without* the use of Holder's inequality leads to (4.30). Note that (4.30) is the same as (4.14), except with β_{qn} in place of α_{qn}. The proof is now completed in the same way as the proof of Theorem 4.3 following (4.14). □

REMARK. The case of (4.30) corresponding to $\nu = 4$, $g(j_1,j_2) = \min\{f(j_1),f(j_2)\}$, and $q = 1$ was in effect established by Révész (1969), as may be seen from a careful scrutiny of the proof of his Theorem MM-3. His method of proof has been utilized. □

The following two corollaries of Theorem 4.5 are immediate. The first result specializes to Conditions B1 and B2. The second result pertains to the case of an infinite sequence $\{X_i\}$.

COROLLARY 4.6. *Let* $X_1,\cdots,X_n$ *satisfy* $b_i^\nu = E\{X_i^\nu\} < \infty$, $1 \le i \le n$, *for an even integer* ν. *Suppose that, for a function* $f(j)$ *and for all* $1 \le i_1 < \cdots < i_\nu \le n$, *either*

$$(4.31) \qquad |E\{X_{i_1}\cdots X_{i_\nu}\}| \le \min\{f(i_2-i_1),f(i_4-i_3),\cdots,f(i_\nu-i_{\nu-1})\}b_{i_1}\cdots b_{i_\nu}$$

or

$$(4.32) \qquad |E\{X_{i_1}\cdots X_{i_\nu}\}| \le f(i_2-i_1)f(i_4-i_3)\cdots f(i_\nu-i_{\nu-1})b_{i_1}\cdots b_{i_\nu}.$$

Let $q \ge 1$ *be given. Put*

$$(4.33) \qquad \beta_{qn}^{(1)} = \left[\frac{1}{(\frac{1}{2}\nu-1)!}\sum_{k=1}^{n} k^{\frac{1}{2}\nu-1} f^q(k)\right]^{\frac{1}{q}}$$

and

$$(4.34) \qquad \beta_{qn}^{(2)} = \left[\frac{1}{(\frac{1}{2}\nu-1)!}\sum_{k=1}^{n}\left(\sum_{j=1}^{k} f^q(j)\right)^{\frac{1}{2}\nu-1} f^q(k)\right]^{\frac{1}{q}}.$$

Define γ *by* (4.7) *and* D_ν *by* (3.9). *Then*

$$(4.35)\qquad E\left\{\left|\sum_{k=1}^{n} c_k X_k\right|^{\nu}\right\} \le [\nu!\,\tilde{\beta}_{qn} + D_\nu]^{\nu/2}\left[\sum_{k=1}^{n} b_k^{\gamma}|c_k|^{\gamma}\right]^{\nu/\gamma},$$

with $\tilde{\beta}_{qn}$ *given by* $\beta_{qn}^{(1)}$ *if* (4.31) *is assumed and by* $\beta_{qn}^{(2)}$ *if* (4.32) *is assumed.*

COROLLARY 4.7. *Suppose that the assumptions of Theorem* 4.5 *or Corollary* 4.6, *for fixed* ν, g, f *and* q, *hold for all* $n = 1,2,\cdots$. *Let* $\tilde{\beta}_q$ *be given by*

$$(4.36)\qquad \tilde{\beta}_q = \left[\frac{1}{(\frac{1}{2}\nu-1)!}\sum_{k=1}^{\infty}\sum_{j_1=1}^{k}\cdots\sum_{j_{\frac{1}{2}\nu-1}=1}^{k} g^q(j_1,\cdots,j_{\frac{1}{2}\nu-1},k)\right]^{\frac{1}{q}}$$

if (4.25) *is assumed, by*

$$(4.37)\qquad \tilde{\beta}_q = \left[\frac{1}{(\frac{1}{2}\nu-1)!}\sum_{k=1}^{\infty} k^{\frac{1}{2}\nu-1} f^q(k)\right]^{\frac{1}{q}}$$

if (4.31) *is assumed, and by*

$$(4.38)\qquad \tilde{\beta}_q = \left[\frac{1}{(\frac{1}{2}\nu-1)!}\left(\sum_{k=1}^{\infty} f^q(k)\right)^{\frac{1}{2}\nu}\right]^{\frac{1}{q}}$$

if (4.32) *is assumed. Then*

$$(4.39)\qquad E\left\{\left|\sum_{k=1}^{n} c_k X_k\right|^{\nu}\right\} \le [\nu!\,\tilde{\beta}_q + D_\nu]^{\nu/2}\left[\sum_{k=1}^{n} b_k^{\gamma}|c_k|^{\gamma}\right]^{\nu/\gamma}.$$

For q = 1, the version of Corollary 4.7 corresponding to Condition B1, i.e.; corresponding to assumption (4.31), has been given by Gapoškin (1972), under the additional condition that $f(\cdot)$ is nonincreasing.

In many typical situations, *both* (4.31) and (4.32) are satisfied, in which case the use of both (4.37) and (4.38) arise as options for consideration. However,

the requirements on $f(\circ)$ for finiteness of $\tilde{\beta}_q$ are milder in the case of (4.38) than in the case of (4.37). Namely, finiteness of $\sum_1^\infty f^q(k)$ instead of $\sum_1^\infty k^{\frac{1}{2}\nu-1} f^q(k)$ is required. For example, consider the stochastic process

$$X(t) = 2^{-\frac{1}{2}}[\xi^2(t) - 1] , \tag{4.40}$$

where $\xi(t)$ is a Gaussian process with $E\{\xi(t)\} \equiv 0$, $E\{\xi^2(t)\} \equiv 1$, and $E\{\xi(t)\xi(t+\tau)\} = R(\tau)$. This physically realizable stochastic process $X(\circ)$ is considered by Magness (1954) for quantitative illustration of non-Gaussianity. Consider the associated discrete-time sequence $\{X_k\}$, where $X_k = X(k)$, $k = 1,2,\cdots$. It is readily seen that

$$|E\{X_{i_1} X_{i_2} X_{i_3} X_{i_4}\}| \le 15|R(i_2-i_1)R(i_4-i_3)| , \tag{4.41}$$

for all $1 \le i_1 <\cdots< i_4$, under the assumption that $R(\tau)$ is nonincreasing as $|\tau|$ increases. Thus $\{X_k\}$ satisfies Condition B1 with $f(j) = \sqrt{15}\,|R(j)|$, provided that $\sum_1^\infty j|R(j)| < \infty$. Also, $\{X_k\}$ satisfies Condition B2 with the same $f(j)$, provided merely that $\sum_1 |R(j)| < \infty$. (Here we have taken $q = 1$.)

A moment inequality for S_n under Condition C will now be presented.

THEOREM 4.8. *Let* $X_1,\cdots,X_n$ *satisfy* $b_i^\nu = E\{X_i^\nu\} < \infty$, $1 \le i \le n$, *for an even integer* ν. *Suppose that, for a function* $f(j)$ *and a symmetric function* g *of* $\frac{1}{2}\nu-1$ *arguments,*

$$|E\{X_{i_1}\cdots X_{i_\nu}\}| \le \min\{f(i_2-i_1), f(i_\nu-i_{\nu-1})\} g(i_3-i_2, i_5-i_4,\cdots, i_\nu-i_{\nu-1}) b_{i_1}\cdots b_{i_\nu} \tag{4.42}$$

for all $1 \le i_1 <\cdots< i_\nu \le n$. *Let* $q \ge 1$ *be given. Put*

$$(4.43)\quad \delta_{qn} = \left[\sum_{\ell=1}^{\frac{1}{2}\nu-1} \sum_{j_\ell=1}^{n} \sum_{j_1=1}^{j_\ell} \cdots \sum_{j_{\ell-1}=1}^{j_\ell} \sum_{j_{\ell+1}=1}^{j_\ell} \cdots \sum_{j_{\frac{1}{2}\nu-1}=1}^{j_\ell} g^q(j_1,\cdots,j_{\frac{1}{2}\nu-1})\right]^{\frac{1}{q}} .$$

Define γ *by* (4.7) *and* D_ν *by* (3.9). *Then*

$$(4.44)\quad E\left\{\left(\sum_{k=1}^{n} c_k X_k\right)^\nu\right\} \le \left[\nu!\ \delta_{qn}\left(\sum_{k=1}^{n} f^q(k)\right)^{\frac{1}{q}} + D_\nu\right]^{\nu/2}\left[\sum_{k=1}^{n} b_k^\gamma |c_k|^\gamma\right]^{\nu/\gamma} .$$

The proof is similar in technique to that of Theorems 4.3 and 4.4 and so is omitted. Likewise, the extension to the case of an infinite sequence is clear.

The next moment inequality for S_n will be derived under the product-moment exchangeable restriction which was discussed in Section 2.

THEOREM 4.9. *Let* $X_1,\cdots,X_n$ *satisfy* $b_i^\nu = E\{X_i^\nu\} < \infty$, $1 \le i \le n$, *for an even integer* ν. *Suppose that, for a constant* $G_{\nu,n}$,

$$(4.45)\quad E\{X_{i_1}\cdots X_{i_\nu}\} = G_{\nu,n}\, b_{i_1}\cdots b_{i_\nu}$$

for all $1 \le i_1 < \cdots < i_\nu \le n$. *Put*

$$(4.46)\quad \Delta_n = |\textstyle\sum_{(\nu)} b_{i_1} c_{i_1} \cdots b_{i_\nu} c_{i_\nu}| .$$

Then

$$(4.47)\quad E\left\{\left(\sum_{k=1}^{n} c_k X_k\right)^\nu\right\} \le \left[|G_{\nu,n}|\Delta_n\left(\sum_{k=1}^{n} b_k^2 c_k^2\right)^{-\nu/2} + D_\nu\right]^{\nu/2}\left(\sum_{k=1}^{n} b_k^2 c_k^2\right)^{\nu/2} .$$

PROOF. Since by the definition of W_ν

$$|E\{W_\nu\}| \le |G_{\nu,n}|\Delta_n ,$$

the proof follows that of Theorem 4.3. $\square$

Condition (4.45) would be satisfied by an exchangeable sequence of random variables. An application of this theorem to rank statistic problems will be presented in Section 5.

The final two results of this section present moment inequalities in which the upper bound on $E\{S_n^{\nu}\}$ is a function of the mixing numbers of the sequence $\{X_i\}$.

CORROLLARY 4.10. *Let* $\{X_i\}$ *be strictly stationary with* $E\{X_i\} \equiv 0$, *strongly mixing, and bounded:* $|X_i| \le C$, *all* i. *Suppose that* $\beta = \sum_1^\infty j\phi_j < \infty$, *where* $\{\phi_j\}$ *are the Rosenblatt mixing numbers. Define* D_4 *by* (3.9). *Then, for all* n,

$$E\left\{\left(\sum_{k=1}^{n} c_k X_k\right)^4\right\} \le C^4[24\beta + D_4]^2\left(\sum_{k=1}^{n} c_k^2\right)^2 . \tag{4.48}$$

The proof follows from Corollary 4.6 since, under the above assumptions, $\{X_i\}$ satisfies Condition B1, as was noted in Section 2. Corollary 4.10 broadens Lemma 20.4 of Billingsley (1968). He obtains essentially the same bounds, but assumes a more stringent mixing condition: in particular, his mixing numbers $\{\phi_j^*\}$ satisfy $\phi_j^* \le \phi_j$. Furthermore, his summability condition on the ϕ_j^*'s is $\sum_1^\infty (\phi_j^*)^{\frac{1}{2}} < \infty$, a stronger restriction than $\sum_1^\infty j\phi_j < \infty$.

THEOREM 4.11. *Let* $\{X_i\}$ *be a strictly stationary sequence with* $E\{X_i\} \equiv 0$ *and bounded:* $|X_i| \le C$, *all* i. *Let* $\phi(I: J)$ *be the mixing numbers for* $\{X_i\}$. *Put*

$$\tau_n = \sum_{k=1}^{n} \sum_{j=1}^{k} \phi(0: k)\, \phi(0: j) . \tag{4.49}$$

and

$$(4.50)\quad \theta_n = \sum_{j_1=1}^{n-3} \sum_{j_2=j_1+1}^{n-2} \sum_{j_3=j_2+1}^{n-1} \min\{\phi(0;j_1,j_2,j_3),\phi(0,j_1;j_2,j_3),\phi(0,j_1,j_2;j_3)\} .$$

Then, for all n,

$$(4.51)\qquad E\left\{\left(\sum_{k=1}^{n} c_k X_k\right)^4\right\} \le C^4[96\{4\tau_n + \theta_n(\textstyle\sum_1^n c_k^4)/(\sum_1^n c_k^2)^2\} + D_4]^2\left(\displaystyle\sum_1^n c_k^2\right)^2 .$$

PROOF. By a lemma of Ibragimov (1962), for $i_1 < i_2 < i_3 < i_4$ and with

$$g(j_1,j_2,j_3) = \min\{\phi(0;j_1,j_2,j_3),\phi(0,j_1;j_2,j_3),\ \phi(0,j_1,j_2;j_3)\} ,$$

$$|E\{X_{i_1}X_{i_2}X_{i_3}X_{i_4}\}| \le 16C^4\ \phi(i_1;i_2)\ \phi(i_3;i_4) + g(i_2-i_1,i_3-i_1,i_4-i_1).$$

Thus

$$|E\{W_4\}| = 4!\ |E\{\sum_{1\le i_1<\cdots<i_4\le n}\cdots\sum c_{i_1}\cdots c_{i_4} X_{i_1}\cdots X_{i_4}\}|$$

$$\le 384C^4 \sum_{1\le i_1<\cdots<i_4\le n}\cdots\sum |c_{i_1}\cdots c_{i_4}|\phi(0,i_2-i_1)\phi(0;i_4-i_3) +$$

$$96C^4 \sum_{1\le i_1<\cdots<i_4\le n}\cdots\sum |c_{i_1}\cdots c_{i_4}|\ g(i_2-i_1,i_3-i_1,i_4-i_1)$$

$$\le\ 96C^4[4\ \tau_n(\textstyle\sum_1^n c_k^2)^2 + \theta_n(\sum_1^n c_k^4)] .$$

The proof is completed by combining this bound with the bound on $E\{Z_4\}$ as was done in the proof of Theorems 4.3 and 4.4. □

In the next section, Theorem 4.11 will be combined with a result of Blum and Rosenblatt (1956) to obtain a central theorem for sums of bounded functions of strongly mixing random variables.

5. Applications of moment inequalities. The first application is concerned with the question of almost sure convergence of an infinite series $\sum_1^\infty c_k X_k$, subject to mild restrictions on the growth of the constants c_i and mild dependence restrictions on the random variables $\{X_i\}$. A consequence of Kolmogorov's classical "three series criterion" is that if the X_i's are mutually independent with 0 means and variances 1 and if the c_i's satisfy $\sum_1^\infty c_i^2 < \infty$, then $\sum_1 c_i X_i$ converges almost surely. If the dependence restriction is reduced in strength to orthogonality, then results due to Rademacher (1922), Menšov (1923), and Tandori (1957) show that the condition $\sum_1^\infty c_i^2 < \infty$ is not strong enough to insure the almost sure convergence of $\sum_1^\infty c_i X_i$. Rademacher and Menšov's results placed the condition $\sum_1^\infty c_i^2 (\log i)^2 < \infty$ on the c_i's in order to obtain the almost sure convergence of $\sum_1^\infty c_i X_i$, where the X_i's are orthogonal with mean 0 and variance 1. Komlós (1972) obtains the almost sure convergence of $\sum_1^\infty c_i X_i$ under the conditions that $\sum_1^\infty c_i^2 < \infty$, and the X_i's are *multiplicative of order* ν, for an even integer $\nu \geq 4$, $E\{X_i^4\} \leq K < \infty$ (all i), $E\{X_i\} \equiv 0$ and $\mathrm{Var}(X_i) \equiv 1$. This result was effectively improved by Gapoškin (1972), who introduced a dependence restriction similar to Condition B1 in place of the multiplicative of order ν assumption. A theorem which allows Condition A, Condition B, or Condition C to replace the multiplicative of order ν restriction in Komlós result will now be proved. Gapoškin's result will be obtained as a corollary to this result. In the proof of the almost sure convergence result, the following maximal inequality will be used.

THEOREM 5.1. (Longnecker and Serfling (1976)). *Let* $Y_1,\cdots,Y_n$ *be arbitrary random variables. Suppose that for constants* $\nu > 0$ *and* $\gamma > 1$, *and for all positive* λ,

$$(5.1)\qquad P\left\{\left|\sum_{k=i}^{j} Y_k\right| \geq \lambda\right\} \leq \lambda^{-\nu}[g(i,j)]^{\gamma} \quad (\textit{all } 1 \leq i \leq j \leq n)\ ,$$

where g *satisfies* $g(i,j) + g(j+1, k) \leq g(i, k)$. *Then for all positive* λ,

$$(5.2)\qquad P\left\{\max_{1\leq i\leq n}\left|\sum_{k=1}^{i} Y_k\right| \geq \lambda\right\} \leq C_{\nu,\gamma}\lambda^{-\nu}[g(1, n)]^{\gamma}\ ,$$

where $C_{\nu,\gamma}$ *is a constant depending on only* ν *and* γ .

With $g(i, j) = K[\sum_{k=i}^{j} b_k^2 c_k^2]$, where K is defined by (4.20), (4.39) or (4.44), Theorems 4.4, 4.7, and 4.8 in conjunction with Chebyshev's inequality demonstrate that condition (5.1) is satisfied with $\gamma = \frac{1}{2}\nu$ and $Y_k = c_k X_k$, where the X_k's satisfy either Condition A, Condition B, or Condition C. Thus, for random variables satisfying any one of the three dependence restrictions Condition A, B, or C, the maximal inequality of Theorem 5.1 is applicable.

THEOREM 5.2. *Let the sequence* $\{X_i\}$ *satisfy, for any even integer* $\nu > 2$, *either Condition* A, *Condition* B, *or Condition* C, *and* $b_i = E\{X_i^{\nu}\} < \infty$ (*all* i). *Then the condition* $\sum_1^{\infty} b_k^2 c_k^2 < \infty$ *implies the almost sure convergence of* $\sum_1^{\infty} c_k X_k$.

PROOF. Assume $\sum_1^{\infty} b_k^2 c_k^2 < \infty$. With $Y_n = \sum_1^n c_k X_k$, it will be shown that $\sum_1^{\infty} c_k X_k$ converges almost surely by showing that the sequence $\{Y_n\}$ is almost surely Cauchy, that is, satisfies

$$P\{|Y_n - Y_m| \to 0 \text{ as } m, n \to \infty\} = 1\ ,$$

or, equivalently,

$$(5.3) \qquad P\left\{\max_{n\geq m}|Y_n - Y_m| > \lambda\right\} \to 0, \text{ as } m \to \infty, \text{ for each } \lambda > 0 .$$

By the remarks following Theorem 5.1, it is seen from (5.2) that

$$(5.4) \qquad P\left\{\max_{m\leq n\leq M}|Y_n - Y_m| > \lambda\right\} \leq \lambda^{-\nu}\theta_\nu\left(\sum_{k=m}^{M} b_k^2c_k^2\right)^{\nu/2} ,$$

where θ_ν does not depend on m and M. If $M \to \infty$ in (5.4), then

$$(5.5) \qquad P\left\{\max_{n\geq m}|Y_n - Y_m| > \lambda\right\} \leq \lambda^{-\nu}\theta_\nu\left[\sum_{k=m}^{\infty} b_k^2c_k^2\right]^{\nu/2} .$$

Since $\sum_1^\infty b_k^2c_k^2 < \infty$, the right-hand side of (5.5) tends to 0 as $m \to \infty$, establishing (5.3). □

Results similar to Theorem 5.2 for random variables satisfying either Condition B1 or Condition B2 follow immediately from Theorem 5.2. The result for Condition B1 is essentially the same as Theorem 3 of Gapǒskin (1972), although he implicitly assumes that f is nonincreasing. In the case of a sequence of random variables satisfying both (2.3a) and (2.4a) of Conditions B1 and B2, the almost sure convergence result for variables satisfying Condition B2 is a more general result than the one for Condition B1. This is evident upon examination of the summability conditions (2.3b) and (2.4b).

In comparing the relative strengths of Gapǒskin's result and Theorem 5.2, it is of interest to examine the case of a stationary Gaussian time series $\{X_k\}$ with $E\{X_k\} \equiv 0$ and $Var(X_k) \equiv 1$. By (2.6) it is easily seen that if $|R(k)|$ is

nonincreasing, then $|E\{X_{i_1}X_{i_2}X_{i_3}X_{i_4}\}| \leq 3\min\{R(i_2-i_1),R(i_4-i_3)\}$. Gapoškin's result would then give that $\sum_1^\infty c_kX_k$ converges almost surely if $\sum_1^\infty c_k^2 < \infty$ and $\sum_1^\infty kR(k) < \infty$. Further examination of (2.6) reveals that this sequence satisfies a combination of Condition B2 and Condition C. Hence, by Theorem 5.2, $\sum_1^\infty c_kX_k$ converges almost surely if $\sum_1^\infty c_k^2 < \infty$ and $\sum_1^\infty R(k) < \infty$. Thus the restriction placed on the covariance function $R(k)$ by Gapoškin's result can be relaxed via Theorem 5.2.

A second area of application of moment inequalities concerns rates of convergence in the central limit theorem for linear rank statistics. Under suitable assumptions, Jurečková and Puri (1975) establish that the rate of convergence of the cumulative distribution function of the simple linear rank statistic

$$S_N = \sum_{i=1}^N c_{Ni}\phi\left(\frac{R_{Ni}}{N+1}\right)$$

to the normal distribution function is $O(N^{-\frac{1}{2}+\delta})$ for any $\delta > 0$, where $C_{N1},\cdots,C_{NN}$ are known constants, $R_{N1},\cdots,R_{NN}$ are the ranks of the independent identically distributed observations $X_{N1},\cdots,X_{NN}$, and $\phi(\cdot)$ is a score generating function. Their technique of proof consists of two main steps, the first of which is to establish the following lemma.

LEMMA (Jurečková and Puri). *Assume that the constants* $C_{N1},\cdots,C_{NN}$ *satisfy* $\sum_1^N C_{Ni} = 0$, $\sum_1^N C_{Ni}^2 = 1$ and $\max\limits_{1\leq i\leq N} C_{Ni}^2 = O(N^{-1}\log N)$. *Let the first derivative of* $\phi(t)$ *exist and be bounded in* $(0,1)$. *Then corresponding to any positive integer* k, *where* $2k+1 < N$, *there exists a constant* $B(k) > 0$ *and a positive integer* N_k *such that for all* $N > N_k$

$$E\{(S_N - T_N)^{2k}\} \leq B(k)N^{-k} ,$$

where $T_N = \sum_1^N C_{Ni}\phi(F(X_i))$ *and* F *is the cdf of* X_1.

The second step is an application of standard results (Loève (1965), p. 288) to obtain the rate of convergence of the cdf of T_N to the normal cdf. These two results are then combined to yield the desired rate of convergence.

The proof of the above lemma is tedious and hence an alternate method of proof is desirable. Since $\{R_{Ni} - \phi(F(X_i))\}$ is an exchangeable sequence and $E\{(R_{Ni} - \phi(F(X_i)))^{2k}\} = O(N^{-k})$, Theorem 4.9 directly yields the desired bound on $E\{(S_N-T_N)^{2k}\}$. Thus the methodology of obtaining this rate of convergence has been simplified since the proof of Theorem 4.9 is more straightforward than Puri and Jurečková's lemma. Moreover, Theorem 4.9 is more general.

A moment inequality plays a major role in proving a central limit theorem for sums of functions of mixing random variables. In Gastwirth and Rubin (1975), a central limit theorem is proved for sums of the form $\sum_1^n f(X_i)$, where f is a bounded function and $\{X_i\}$ is a strongly mixing stationary sequence. The following theorem broadens their result.

THEOREM 5.3. *Let* $\{X_i\}$ *be a strongly mixing stationary sequence. Suppose that the mixing numbers of* $\{X_i\}$ *satisfy*

$$\sum_{k=1}^{\infty} \phi(0;k) < \infty \tag{5.6}$$

and

$$\sum_{j_1=1}^{n-3} \sum_{j_2=j_1+1}^{n-2} \sum_{j_3=j_2+1}^{n-1} \min\{\phi(0;j_1,j_2,j_3),\phi(0,j_1;j_2,j_3),\phi(0,j_1,j_2;j_3)\} = O(n) . \tag{5.7}$$

Then any random variable of the form $S_n = \sum_1^n f(X_i)$, *where* f *is a bounded function, is asymptotically normally distributed, that is,*

(5.8) $$n^{-\frac{1}{2}}[S_n - E\{S_n\}] \xrightarrow{\mathcal{D}} N(0, \sigma^2) ,$$

where $\sigma^2 = \lim_{n\to\infty} n^{-1}\mathrm{Var}\{S_n\}$.

PROOF. Let $Y_i = f(X_i) - E\{f(X_i)\}$ and let K be a constant such that $|f(x)| \le \frac{1}{2}K$ for all real x. With $c_i \equiv 1$, Theorem 4.11 implies that

$$E\left\{\left(\sum_{i=1}^{n} Y_i\right)^4\right\} \le K^4[96(4\tau_n + n^{-1}\theta_n) + D_4]^2 n^2 ,$$

where τ_n and θ_n are defined by (3.49) and (3.50) respectively. By (5.7), $\theta_n = O(n)$ and since $\tau_n \le (\sum_1^\infty \phi(0; k))^2 < \infty$, it follows that

(5.9) $$E\{(\textstyle\sum_1^n Y_i)^4\} = O(n^2) .$$

Furthermore, (5.6), the stationarity of $\{Y_i\}$ and the condition $|Y_i| \le K$ immediately imply that

(5.10) $$E\{(\textstyle\sum_1^n Y_i)^2\} \sim h(n) \quad \text{as } n \to \infty ,$$

where $h(n) = n(E\{Y_0^2\} + 2\sum_1^\infty E\{Y_oY_k\})$. By (5.9) and (5.10), the conditions of the Blum-Rosenblatt (1956) theorem hold and hence (5.8) follows. □

The method of proof of Gastwirth and Rubin (1975) has been simplified. Also Theorem 5.3 slightly relaxes their conditions on the mixing numbers since they require

$$\sum_{j\neq k} \sum_{1\le j+k\le n} \min\{\phi(0,j;k),\phi(0;j,k),\phi(0,k;j)\} = O(n)$$

along with restrictions (5.6) and (5.7). Since the conditions of Theorem 5.3 hold

whenever $\sum_1^n k^2\phi(0; k) = O(n)$, the calculations of Gastwirth and Rubin demonstrate that the double-exponential, the Gaussian Markov, and the Cauchy processes all satisfy the conditions of Theorem 5.3. (These processes also satisfy the conditions of the Gastwirth-Rubin Theorem.)

REFERENCES

ANDERSON, T. W. (1971). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

BLUM, J. R. and ROSENBLATT, M. (1956). A class of stationary processes and a central limit theorem. *Proc. Nat. Acad. Sci.* 43 412-423.

BURNSIDE, W. S. and PANTON, A. W. (1899). *The Theory of Equations,* Vol. I, 4th ed. Dublin University Press, Dublin.

GAPOŠKIN, V. F. (1972). On the convergence of series of weakly multiplicative systems of functions. *Mathematics of the USSR* 18 361-372.

GASTWIRTH, J. L. and RUBIN, H. (1975). The asymptotic distribution theory of the empiric cdf for mixing stochastic processes. *Ann. Statist.* 3 809-824.

IBRAGIMOV, I. A. (1962). A central limit theorem for a class of dependent random variables. *Theor. Prob. Applic.* 8 83-89.

JUREČKOVÁ, J. and PURI, M. L. (1975). Order of normal approximation for rank test statistics distribution. *Ann. Prob.* 3 526-533.

KAC, M., SALEM, R. and ZYGMUND, A. (1948). A gap theorem. *Trans. Amer. Math. Soc.* 63 235-248.

KOMLÓS, J. (1972). On the series $\sum c_k\phi_k$. *Stud. Sci. Math. Hung.* 7 451-458.

LOÈVE, M. (1963). *Probability Theory.* Van Nostrand, Princeton.

LONGNECKER, M. and SERFLING, R. J. (1976). General moment and probability inequalities for the maximum partial sum. *Acta Math. Acad. Sci. Hungar.* (to appear).

MAGNESS, T. A. (1954). Spectral response of a quadratic device to non-Gaussian noise. *J. Appl. Phys.* 25 1357-1365.

MENŠOV, D. (1923). Sur les séries de fonctions orthogonales, I. *Fund. Math.* 4 82-105.

MÓRICZ, F. (1976). On the convergence properties of weakly multiplicative systems. *Acta Sci. Math.* 38 127-144.

RADEMACHER, H. (1922). Einige Sätze über Reihen von allgemeinen Orthogonalfunctionen. *Math. Ann.* 87 112-138.

RÉVÉSZ, P. (1969). M-mixing systems. I. *Acta Math. Acad. Sci. Hung.* 20 431-442.

SERFLING, R. J. (1969). Probability inequalities and convergence properties for sums of multiplicative random variables. Florida State University Statistics Report M151.

TANDORI, K. (1957). Über die orthogonalen Funktionen, I. *Acta Sci. Math.* 18 57-130.

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MOMENT INEQUALITIES FOR S_n UNDER GENERAL DEPENDENCE

RESTRICTIONS, WITH APPLICATIONS

Consider ~~the~~ a sum $S_n = \sum_{k=1}^{n} c_k X_k$, where $\{X_i\}$ is composed of a sequence of random variables and $\{c_i\}$ a sequence of constants. This paper establishes moment inequalities ~~of~~ with the form $E\{S_n^{\nu}\} \le A(\sum_{k=1}^{n} b_k^r c_k^r)^{\nu/r}$, where ν is an even integer, $b_k = E\{X_k^{\nu}\}$ $(k=1,\dots,n)$, and A is a constant depending upon ν and the dependence restrictions imposed upon the ~~$\{X_i\}$~~ random variables but not depending upon the ~~$\{c_i\}$~~ constants. A further inequality of more complicated form is also established. The dependence restrictions considered are either of the weak multiplicative type or of related types, namely exchangeable sequences and strongly mixing sequences. Three applications are developed. One treats the almost sure convergence of series $\sum_{k=1}^{\infty} c_k X_k$, under mild dependence restrictions and ~~the~~ finite limit condition $\sum_{k=1}^{\infty} c_k^2 \cancel{< \infty}$. Secondly, an improved technique is presented for the problem of establishing the rate of convergence in the central limit theorem for simple linear rank statistics. Finally, the central limit theorem for strongly mixing summands is treated.